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Mathematical modelling of boundary conditions for magneto-sensitive elastomers: variational formulations

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Abstract The mathematical modelling of the behaviour of magneto-sensitive (MS) elastomers is complicated due to the strong nonlinear coupling of the mechanical and electromagnetic responses. An additional problem is that one needs to consider the surrounding space in the calculation of the electromagnetic fields. Finding appropriate and simple variational formulations is an important step towards solving the governing equations using numerical methods. Variational formulations for a number of boundary-value problems are given, and different possible models for the mechanical and magnetic boundary conditions are studied.

Keywords Magneto-elasticity · Nonlinear elasticity · Smart materials · Variational formulation

1 Introduction

During the last years there has been a growing interest in studying the theory of magneto– and electro–elasticity with finite deformations [1]. One of the main applications of this theory is the mathematical modelling of magneto- and electro-sensitive elastomers, which has been studied in detail by, for example, Dorfmann and Ogden [2–6], Kankanala and Triantafyllidis [7] and Steigmann [8]. Magento- and electro-sensitive elastomers are considered very interesting materials, because their properties can be tuned by applying an external magnetic or electric field [9–13].

In the previous research we can see that the governing equations are too complicated to solve any, albeit very simple boundary-value problems with exact methods. Therefore, it is very important to develop numerical methods to solve these boundary-value problems. The finite-element method has been found to be one of the most suitable for this problem. A first step towards the implementation of the finite-element method is to find appropriate variational formulations; this had been done, for example, by Bustamante et al. [14,15] for the magneto–elastic and the electro–elastic cases, respectively.

The main problem to be discussed in this paper corresponds to the boundary conditions in magneto–elasticity. We explore different models for the interaction of magneto-sensitive elastomeric bodies with an external *loading device*; for these models we propose variational formulations, which not only consider the body, but also

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the surrounding free space and other external bodies. In these variational formulations a careful distinction has been made between fields and quantities that are truly external data, such as traction or distant magnetic fields, and the fields and quantities that are unknown, and must be determined from the solution of the boundary-value problem.

The equations of electromagnetism must hold everywhere, not only inside the MS elastomeric body, but also in the surrounding areas. In magneto– and electro–elasticity, as a first approximation, some researchers have only considered the material body in their analyses and have not accounted, for example, for the free space surrounding the body (see, for example, [16]). The same has happened in the classical theory of linear electro– and magneto–elasticity (see, for example, [17–20]).

Other authors have considered the case of a material body completely surrounded by free space [7,21,22]. In electro–elasticity an interesting boundary condition corresponds to an electric potential or an electric distribution of charges, which is applied to a portion of the surface of the body; see, for example, [16,23–25].

In the classical theory of elasticity there are two important boundary conditions, which have been used for most problems, namely a prescribed external surface stress or traction, and a prescribed displacement on the boundary of a body. The problem where we prescribe these two boundary conditions on disjoint surfaces is called the mixed-boundary-condition problem. It is necessary to point out that these two conditions do not include, by any means, all of the possible boundary conditions that can be found in *real* problems; additional examples of boundary conditions can be found in [26, Sects. 5.1 and 5.2], and [27].

When saying that we apply a prescribed surface traction or restriction on the displacement, what we mean is that the body is in contact with an external body (or *loading device* using Batra's terminology [28]), and we can assume that the result of this surface contact can be approximated with a model, where we prescribe external surface forces and a restriction in the displacement. However, in the *real* situation what happens is that this external *loading device* is also affected by the interaction with the body, and this interaction will have an effect on the external surface force and on the prescribed displacement. In fact, as shown by Batra [28], in a *real* problem we are not able to decompose so easily the interaction in surface forces and given displacements; moreover, the interaction is not *local*, in the sense that, for example, the interacting force at each point depends on the deformation at every point of the surface. Therefore, it is necessary to point out that the problem of modelling boundary conditions in elasticity is not simple, and there are still open questions (see, for example, [29,30]).

If the problem of modelling boundary conditions in the classical theory of elasticity is complicated, we can expect that the situation in magneto– and electro–elasticity will be even more difficult. Even if we consider a simplified model (as we will see in detail in Sect. 3), besides the prescription of a surface mechanical load and a displacement, we can (for example, for the electro-elastic case) prescribe a given electric potential and an electric displacement on disjoint portions of the surface of the body (see, for example, [16]); but to assume that the electric displacement is given on the surface of the body, means we are not concerned with the behaviour of the fields outside the body. However, for example, in the magneto–elastic problem, the numerical results of Bustamante et al. [21], for the behaviour of a MS tube surrounded by free space, suggest that it may be actually important to consider the surrounding space for the calculation of these external fields.

In magneto– and electro–elasticity several authors have considered the problem of a body surrounded completely by free (vacuum) space (see, for example, [1,7,8,14,15,22]). But the question remains: what should we do when we have external mechanical surface forces and restrictions on the displacement? To arrive at an appropriate and simple model for the direct contact of bodies in magneto– and electro–elasticity is not clear. In [9] experimental results were obtained for MS elastomers by attaching cylindrical specimens directly to a traction device; other experiments involving the interaction between mechanical and magnetic or electric effects are described in [10,13,31]. The incorporation of *Maxwell stresses* has also caused some controversy; see [32,33].

The development in the subsequent sections will be restricted to MS systems only. In Sect. 2 we provide an overview of the theory for MS elastomers developed by Dorfmann and Ogden [2–6]. In Sect. 3 we discuss different models for boundary conditions, and in Sect. 4 we propose variational formulations for some of the above models. In Sect. 5 we include some final remarks. Some of the results presented in this paper are partly based on the results presented in [34].

2 Basic equations

2.1 Kinematics

We work with a deformable magnetically sensitive body that is initially in an unstressed configuration, the region occupied by the body in this reference configuration being denoted $\mathcal{B}_r$, with boundary $\partial\mathcal{B}_r$. The position vector of a material point within the body in this configuration is denoted $\mathbf{X}$. The material is deformed from the configuration $\mathcal{B}_r$ so that the point $\mathbf{X}$ occupies the position $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ in the deformed configuration, denoted by $\mathcal{B}$, which has boundary $\partial\mathcal{B}$.

In this section we assume that the body is completely surrounded by an infinitely large free (vacuum) space[1] denoted $\mathcal{B}'$, far away its boundary is denoted $\partial\mathcal{B}^\infty$ and its boundary with the body is denoted $\partial\mathcal{B}'$. We note that, if $\mathbf{n}$ and $\mathbf{n}'$ are the normal unit (outward) vectors to $\partial\mathcal{B}$ and $\partial\mathcal{B}'$, respectively, then $\mathbf{n} = -\mathbf{n}'$. Only the Eulerian description of this exterior free space makes sense from a physical point of view; however, it is possible to extend the field $\boldsymbol{\chi}$ outside the body [22].

The deformation gradient tensor $\mathbf{F}$ relative to $\mathcal{B}_r$ is defined by [35]

$$\mathbf{F} = \operatorname{Grad}\boldsymbol{\chi}, \quad \mathbf{X} \in \mathcal{B}_r, \tag{1}$$

where Grad denotes the gradient operator with respect to $\mathbf{X}$. We adopt the notation

$$J = \det\mathbf{F} \tag{2}$$

with the standard convention $J > 0$.

The right and left Cauchy–Green tensors, denoted $\mathbf{c}$ and $\mathbf{b}$, respectively, are defined by

$$\mathbf{c} = \mathbf{F}^{\mathrm{T}}\mathbf{F}, \quad \mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}, \tag{3}$$

where ${}^{\mathrm{T}}$ denotes the transpose of a second-order tensor.

For a detailed discussion of the relevant background in nonlinear elasticity and continuum mechanics we refer to [35].

2.2 The equations of magnetostatics

Consider that the deformed configuration $\mathcal{B}$ arises from the application of mechanical loads and a magnetic field. We denote by $\mathbf{H}$, $\mathbf{B}$ and $\mathbf{M}$, respectively, the magnetic field, the magnetic induction and the magnetization in this configuration. These are Eulerian vector fields, i.e., they are defined on the deformed body $\mathcal{B}$ and are regarded as functions of $\mathbf{x}$.

The fields $\mathbf{H}$ and $\mathbf{B}$ satisfy the field equations [36]

$$\operatorname{curl}\ \mathbf{H} = \mathbf{0}, \quad \operatorname{div}\ \mathbf{B} = 0, \tag{4}$$

which are the appropriate specializations of Maxwell's equations in the absence of distributed currents and time dependence. The magnetization vector can be considered as a derived quantity and is defined in terms of $\mathbf{H}$ and $\mathbf{B}$ by the standard equation (see [36])

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}). \tag{5}$$

Equation (5) gives an explicit expression for $\mathbf{M}$ in terms of either $\mathbf{H}$ or $\mathbf{B}$ as the independent variable. In vacuum or in non-magnetizable materials, $\mathbf{M} = \mathbf{0}$ and (5) reduces to

$$\mathbf{B} = \mu_0\mathbf{H}. \tag{6}$$

[1] In the subsequent sections we assume other types of boundary conditions.

Across a surface of discontinuity in the body or across the boundary $\partial\mathcal{B}$, the jump conditions for the fields are

$$\mathbf{n} \times [\![\mathbf{H}]\!] = \mathbf{0}, \quad \mathbf{n} \cdot [\![\mathbf{B}]\!] = 0 \quad \text{on} \quad \partial\mathcal{B}, \tag{7}$$

where the double square brackets means, for example, for the magnetic field that $[\![\mathbf{H}]\!] \equiv \mathbf{H}^{\mathrm{o}} - \mathbf{H}^{\mathrm{i}}$, where $\mathbf{H}^{\mathrm{o}}$ and $\mathbf{H}^{\mathrm{i}}$ correspond to the magnetic field evaluated outside and inside the body across $\partial\mathcal{B}$, respectively. The vector $\mathbf{n}$ is the unit outward normal to $\partial\mathcal{B}$.

Using (5), (6) and (7) we may easily show that

$$[\![\mathbf{H}]\!] = (\mathbf{n} \cdot \mathbf{M})\mathbf{n}, \quad [\![\mathbf{B}]\!] = \mu_0((\mathbf{n} \cdot \mathbf{M})\mathbf{n} - \mathbf{M}) \quad \text{on} \quad \partial\mathcal{B}. \tag{8}$$

2.3 Mechanical balance laws

If we denote by ρ_r and ρ the mass densities in the reference and current configurations, respectively, then the conservation-of-mass equation is given by

$$J\rho = \rho_r. \tag{9}$$

We can write the magnetic body forces as the divergence of a second-order tensor and add this tensor to the Cauchy stress tensor to define a 'total (Cauchy) stress tensor', which we denote by $\boldsymbol{\tau}$ (see, for example, [3,5]). In such a case the equilibrium equation in the absence of mechanical body forces can be written in the form

$$\operatorname{div} \boldsymbol{\tau} = \mathbf{0}. \tag{10}$$

Since the magnetic body forces and couples are included in the definition of the stress, we have $\boldsymbol{\tau}^{\mathrm{T}} = \boldsymbol{\tau}$ from a balance of angular momentum.

The counterpart of the nominal stress tensor is denoted here by $\mathbf{T}$, and is defined by

$$\mathbf{T} = J\mathbf{F}^{-1}\boldsymbol{\tau}; \tag{11}$$

the equilibrium equation (10) can be written in the alternative form

$$\operatorname{Div} \mathbf{T} = \mathbf{0}. \tag{12}$$

2.4 The formulation of Dorfmann and Ogden

We base our research on the formulation developed by Dorfmann and Ogden [2–6]. The formulation of Dorfmann and Ogden [2–4] for magneto–elastic materials is based on the assumption that there exists a total energy function, here denoted $\Omega = \Omega(\mathbf{F}, \mathbf{B}_l)$, such that

$$\mathbf{T} = \frac{\partial \Omega}{\partial \mathbf{F}}, \quad \mathbf{H}_l = \frac{\partial \Omega}{\partial \mathbf{B}_l}, \tag{13}$$

where

$$\mathbf{B}_l = J\mathbf{F}^{-1}\mathbf{B}, \quad \mathbf{H}_l = \mathbf{F}^T\mathbf{H} \tag{14}$$

are the Lagrangian forms of the magnetic induction and the magnetic field vectors, respectively.

On use of the partial Legendre transformation we can define the complementary energy function

$$\Omega^* = \Omega^*(\mathbf{F}, \mathbf{H}_l) = \Omega(\mathbf{F}, \mathbf{B}_l) - \mathbf{H}_l \cdot \mathbf{B}_l \tag{15}$$

such that

$$\mathbf{T} = \frac{\partial \Omega^*}{\partial \mathbf{F}}, \quad \mathbf{B}_l = -\frac{\partial \Omega^*}{\partial \mathbf{H}_l}. \tag{16}$$

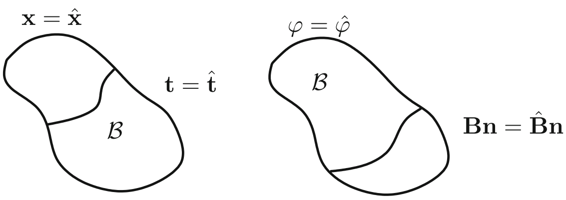

Fig. 1 Boundary conditions for the magneto–elastic problem considering only the material body $\mathcal{B}$

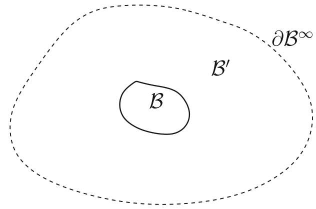

Fig. 2 Sketch of a body $\mathcal{B}$ surrounded by free space $\mathcal{B}'$ with boundary $\partial\mathcal{B}^\infty$

3 Boundary conditions for mixed boundary-value problems

In the classical linear theory of magneto–elasticity and electro– (piezo–) elasticity only the material body has been considered in the analysis; see, for example, [16,18–20]. The same has been done in the more general case where we work with finite deformations; see, for example, [16,24,37]. In the magneto–elastic problem the surface $\partial\mathcal{B}$ of the body $\mathcal{B}$ is divided, for example, in two different forms (see Fig. 1)

$$\partial\mathcal{B}=\partial\mathcal{B}_{\mathbf{x}}\cup\partial\mathcal{B}_{\mathbf{t}},\quad \partial\mathcal{B}=\partial\mathcal{B}_{\varphi}\cup\partial\mathcal{B}_{\mathbf{B}}, \tag{17}$$

such that

$$\partial\mathcal{B}_{\mathbf{x}}\cap\partial\mathcal{B}_{\mathbf{t}}=\emptyset,\quad \partial\mathcal{B}_{\varphi}\cap\partial\mathcal{B}_{\mathbf{B}}=\emptyset, \tag{18}$$

where $\partial\mathcal{B}_{\mathbf{x}}$ is the part of $\partial\mathcal{B}$ where $\mathbf{x}=\hat{\mathbf{x}}$ is prescribed, $\partial\mathcal{B}_{\mathbf{t}}$ is the part of $\partial\mathcal{B}$ where an external surface load $\mathbf{t}=\hat{\mathbf{t}}$ is given, $\partial\mathcal{B}_{\varphi}$ is the part of $\partial\mathcal{B}$ with a known external magnetic potential $\varphi=\hat{\varphi}$, and $\partial\mathcal{B}_{\mathbf{B}}$ is the part where the magnetic induction is given $\mathbf{Bn}=\hat{\mathbf{B}}\mathbf{n}$.

There are some problems with the above model. First, as pointed out, for example, by Bustamante et al. [15], if the above model does not take into account the surrounding space, then Gauss's law, which must hold everywhere, would be violated. So, what happens is that there is a magnetic interaction of the body with the surrounding space, which means that prescribed values of, for example, $\hat{\varphi}$ and $\hat{\mathbf{B}}$ depend on this interaction and must be calculated rather than given as data. In magneto–elasticity the results obtained by Bustamante et al. [21] suggest that considering the surrounding space may have an important effect on the distribution and magnitude of this external field. Therefore, when we assume an isolate body and we give $\hat{\varphi}$ and $\hat{\mathbf{B}}$ as data, these values for the external potential and magnetic induction are only *approximate* quantities.

A second problem with the above mode is that there is no discussion about how an external mechanical surface load (or a restriction on the displacement) could be applied. As mentioned in the Introduction, surface interaction means we are assuming that the body is interacting with another external body.

A better approximation for the problem is to consider a body $\mathcal{B}$ completely surrounded by free (vacuum) space $\mathcal{B}'$ (see Fig. 2), assuming that $\mathcal{B}'$ extends to infinity (see, for example, [7,8,14,15,22]).

In this model the surrounding free space is considered in the analysis. It is assumed that some external field (or potential) is applied far away at the external boundary of $\mathcal{B}'$, which we denote by $\partial\mathcal{B}^\infty$. If we work with the total stress, it is possible to show that, besides any mechanical surface traction, we have to include as external load the traction due to the *Maxwell stresses* (see [33]).

In the above model the electromagnetic laws hold everywhere, but still no deeper analysis has been provided about how an external mechanical surface traction or restriction on the displacement can be applied. We need to propose some simple models for the interaction of the body $\mathcal{B}$ with an external *loading device*, where we can assume that this *loading device* is semi-infinite.

We now explore two possible models for the interaction of a MS body and an external *loading device*. Regarding magnetic fields, there are basically two methods to generate them: one method is to use an electric current in an electromagnet that generates a magnetic induction $\mathbf{B}$, a second alternative is to use a permanent magnet (such

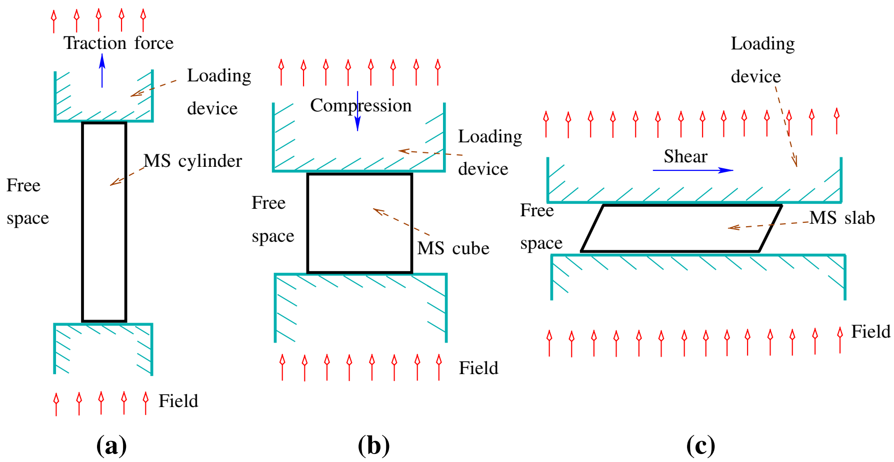

Fig. 3 Schemes for three different experiment for MS elastomers

as loadstone), which generates a magnetic field **H**. As regards the application of a mechanical surface load or a restriction on the displacement, consider Fig. 3. We have schematic representations of three possible experiments, Fig. 3a shows the uniform extension of a MS cylinder (see [9]), Fig. 3b shows the compression of a MS cube [38], and Fig. 3c shows the simple shear of a MS slab (see [13]).

If we assume that an external field is applied far away (either on the external boundary of the free space, or far away on the external boundary of the loading device) two questions remain:

- What are the properties of this *loading device*?
- What kind of interaction do we have in the interface between a MS elastomer and a loading device?

In his paper on the boundary conditions in the context of the classical theory of nonlinear elasticity, Batra [28] assumed the loading device as a deformable body. In our case, we restrict ourselves to the situation, where the MS elastomeric body interacts with an external device made up of a material with a much higher 'stiffness' than the MS elastomer. In principle, we assume the external device as rigid, but we relax this assumption in Sect. 4.3.

Regarding the second question about the interaction on the interface, from Fig. 3a, c we assume that the MS elastomer is perfectly bonded to the loading device. For the problem presented in Fig. 3b we may have, for example, the situation where the surface of the cube is not perfectly bonded but just in contact with the device; we could even assume there is no friction on the interface, and so the surface of the cube would be free to slip in the tangential direction to the compression. For the sake of simplicity, we only consider the case where the MS elastomer is totally bonded to the *loading device*.

We consider two models:

First model: The MS body is bonded to a loading device that is rigid, magnetizable and semi-infinite. Part of the surface of the MS body is in contact with the free space; the loading device may rotate and move rigidly. Far away on both the free space and the loading device there is an external applied field.

Second model: In the second case we consider an MS body attached to two devices, one of them being rigid and the other deformable (elastic), while both devices are magnetizable.

More details of these models are provided in Sect. 4.

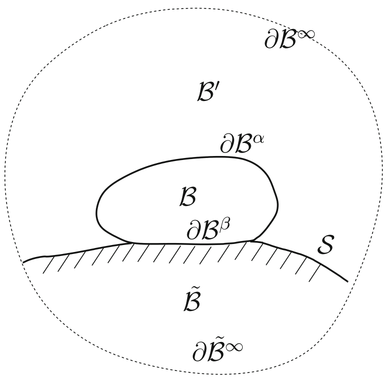

Fig. 4 A mixed-boundary-value problem

4 Variational formulations

4.1 Interaction of a MS elastomer with a rigid semi-infinite body. Formulation based on the scalar magnetic potential

Consider Fig. 4 (see [34]), which shows a body $\mathcal{B}$ with part of its boundary bonded to a semi-infinite rigid body (*loading device*) $\tilde{\mathcal{B}}$.

The body $\tilde{\mathcal{B}}$ may displace and rotate. Exterior to these bodies is the free space $\mathcal{B}'$. The surface of $\mathcal{B}$ is divided into two disjoint and complementary parts, $\partial\mathcal{B} = \partial\mathcal{B}^{\alpha} \cup \partial\mathcal{B}^{\beta}$, where $\partial\mathcal{B}^{\alpha}$ adjoins the free space and $\partial\mathcal{B}^{\beta}$ is in contact with the surface of $\tilde{\mathcal{B}}$. The body $\tilde{\mathcal{B}}$ and the free space $\mathcal{B}'$ are separated by the surface $\mathcal{S}$. The boundaries of $\mathcal{B}'$ and $\tilde{\mathcal{B}}$ far away are denoted by $\partial\mathcal{B}^{\infty}$ and $\partial\tilde{\mathcal{B}}^{\infty}$, respectively. The normal vectors on the boundaries of $\mathcal{B}$, $\tilde{\mathcal{B}}$ and $\mathcal{B}'$ are denoted by $\mathbf{n}$, $\tilde{\mathbf{n}}$ and $\mathbf{n}'$, respectively, directed outwards from the region in each case; note that, for example, on $\partial\mathcal{B}^{\alpha}$ we have $-\mathbf{n}' = \mathbf{n}$, and on $\mathcal{S}$ we have $-\tilde{\mathbf{n}} = \mathbf{n}'$.

The body $\mathcal{B}$ is magneto–elastic. We work first with the magnetic field as the independent magnetic variable [3,4], and so, the free-energy function for $\mathcal{B}$ depends on the deformation gradient and the magnetic field. We assume that the rigid body $\tilde{\mathcal{B}}$ is magnetizable, and that the energy function depends only on the magnetic field $\mathbf{H}$.

For the boundary conditions on $\partial\mathcal{B}$ we prescribe a displacement on $\partial\mathcal{B}^{\beta}$ (the rigid displacement of $\tilde{\mathcal{B}}$), and we denote its position vector by $\hat{\mathbf{x}}$, while the boundary $\partial\mathcal{B}^{\alpha}$ is taken to be free of mechanical traction. We also apply an external magnetic induction far away on $\partial\mathcal{B}^{\infty}$ and/or $\partial\tilde{\mathcal{B}}^{\infty}$.

The above model may be used to describe some real experiments, such as the uniaxial tension of a bar [9] and the shear of a slab [13]. However, here what is *given* as an external datum is the rigid displacement and rotation of $\tilde{\mathcal{B}}$, not the *external* (far away) applied force.

Consider the expression

$$E\{\mathbf{x}, \varphi\} = \int_{\mathcal{B}\cup\tilde{\mathcal{B}}} \varrho\Upsilon \,\mathrm{d}v + \frac{1}{2}\int_{\mathcal{B}\cup\tilde{\mathcal{B}}} \mathbf{B}\cdot\mathbf{H}\,\mathrm{d}v - \frac{1}{2}\mu_0\int_{\mathcal{B}\cup\tilde{\mathcal{B}}} \mathbf{M}\cdot\mathbf{H}_a\,\mathrm{d}v \tag{19}$$

for the energy of the bodies $\mathcal{B}$ and $\tilde{\mathcal{B}}$, where ϱ is the mass density and Υ the energy per unit mass. This is a modification of the energy used in [14] and is based on the classical formulation of Brown [39]. In the latter formulation the fields $\mathbf{H}$ and $\mathbf{B}$ are each decomposed as the sum of an *applied* field in the absence of material and an additional *self* field generated by the presence of the magnetic material body. We use the subscripts a and s, respectively, to refer to these two parts. For more details, see [8,14].

A solution of the field equation $(4)_1$ is given in terms of the magnetic scalar potential[2] φ, such that

$$\mathbf{H} = -\mathrm{grad}\,\varphi. \tag{20}$$

[2] This solution is valid only for the case there is no electric interaction and no time dependence.

This potential is also decomposed into two parts and we write

$$\varphi = \varphi_a + \varphi_s, \tag{21}$$

where

$$\mathbf{H}_a = -\mathrm{grad}\,\varphi_a \quad \text{and} \quad \mathbf{H}_s = -\mathrm{grad}\,\varphi_s. \tag{22}$$

The energy function appearing in (19) is defined as

$$\Upsilon = \begin{cases} \psi^* & \mathbf{x} \in \mathcal{B}, \\ \tilde{\psi}^* & \mathbf{x} \in \tilde{\mathcal{B}}, \end{cases} \tag{23}$$

where $\psi^* = \psi^*(\mathbf{F}, \mathbf{H})$ and $\tilde{\psi}^* = \tilde{\psi}^*(\mathbf{H})$. As well as this, for the density ϱ we have

$$\varrho = \begin{cases} \rho & \mathbf{x} \in \mathcal{B}, \\ \tilde{\rho} & \mathbf{x} \in \tilde{\mathcal{B}}, \end{cases} \tag{24}$$

where ρ and $\tilde{\rho}$ are the mass densities associated with $\mathcal{B}$ and $\tilde{\mathcal{B}}$, respectively.

The first term of the right side of (19) can be decomposed as

$$\int_{\mathcal{B}} \rho \psi^* \,\mathrm{d}v + \int_{\tilde{\mathcal{B}}} \tilde{\rho} \tilde{\psi}^* \,\mathrm{d}v. \tag{25}$$

For the second integral in (19), on use of (20), $(4)_2$ and the divergence theorem, we obtain [14]

$$-\frac{1}{2}\int_{\partial\mathcal{B}^\alpha \cup \mathcal{S}} \varphi\, \mathbf{B}\cdot\mathbf{n}\,\mathrm{d}a - \frac{1}{2}\int_{\partial\tilde{\mathcal{B}}^\infty} \varphi\, \mathbf{B}\cdot\tilde{\mathbf{n}}\,\mathrm{d}a. \tag{26}$$

The first integral in the above expression can be replaced by

$$-\frac{1}{2}\mu_0 \int_{\mathcal{B}'} \mathbf{H}\cdot\mathbf{H}\,\mathrm{d}v - \frac{1}{2}\int_{\partial\mathcal{B}^\infty} \varphi\,\mathbf{B}\cdot\mathbf{n}'\,\mathrm{d}a, \tag{27}$$

where again we have used the divergence theorem and the relation (6) appropriate for the free space $\mathcal{B}'$.

The third integral in (19), on use of (5) and the fact that $\mathbf{B}_a = \mu_0 \mathbf{H}_a$ for the whole space $\mathcal{B}\cup\tilde{\mathcal{B}}\cup\mathcal{B}'$, can be rewritten as

$$-\frac{1}{2}\int_{\mathcal{B}\cup\tilde{\mathcal{B}}} \mathbf{B}\cdot\mathbf{H}_a\,\mathrm{d}v + \frac{1}{2}\int_{\mathcal{B}\cup\tilde{\mathcal{B}}} \mathbf{H}\cdot\mathbf{B}_a\,\mathrm{d}v. \tag{28}$$

Following a similar procedure as for the second integral in (19), on use of the divergence theorem, and with reference to Fig. 4, we may rewrite the first term in the above equation as

$$\frac{1}{2}\int_{\partial\mathcal{B}^\alpha\cup\mathcal{S}} \varphi_a\,\mathbf{B}\cdot\mathbf{n}\,\mathrm{d}a + \frac{1}{2}\int_{\partial\tilde{\mathcal{B}}^\infty} \varphi_a\,\mathbf{B}\cdot\tilde{\mathbf{n}}\,\mathrm{d}a, \tag{29}$$

and the first term in (29) is equivalent to

$$\frac{1}{2}\mu_0\int_{\mathcal{B}'} \mathbf{H}\cdot\mathbf{H}_a\,\mathrm{d}v + \frac{1}{2}\int_{\partial\mathcal{B}^\infty} \varphi_a\,\mathbf{B}\cdot\mathbf{n}'\,\mathrm{d}a. \tag{30}$$

The second integral in (30) is easily shown to be equal to

$$-\frac{1}{2}\mu_0\int_{\mathcal{B}'} \mathbf{H}_a\cdot\mathbf{H}\,\mathrm{d}v - \frac{1}{2}\int_{\partial\mathcal{B}^\infty} \varphi\,\mathbf{B}_a\cdot\mathbf{n}'\,\mathrm{d}a - \frac{1}{2}\int_{\partial\tilde{\mathcal{B}}^\infty} \varphi\,\mathbf{B}_a\cdot\tilde{\mathbf{n}}\,\mathrm{d}a. \tag{31}$$

Using (29), (30) and (31) in (28), and (27) in (26), and then combining these results in (19) we can express E as

$$E = \int_{\mathcal{B}} \rho\psi^*\,\mathrm{d}v + \int_{\tilde{\mathcal{B}}} \tilde{\rho}\tilde{\psi}^*\,\mathrm{d}v - \frac{1}{2}\mu_0\int_{\mathcal{B}'} \mathbf{H}\cdot\mathbf{H}\,\mathrm{d}v - \frac{1}{2}\int_{\partial\mathcal{B}^\infty\cup\partial\tilde{\mathcal{B}}^\infty} (\varphi_s\,\mathbf{B} + \varphi\,\mathbf{B}_a)\cdot\mathbf{n}'\,\mathrm{d}a, \tag{32}$$

where, for brevity, we have written $\tilde{\mathbf{n}} = \mathbf{n}'$ on $\partial\tilde{\mathcal{B}}^\infty$. The latter integral in (32) can be decomposed as

$$-\int_{\partial\mathcal{B}^\infty\cup\partial\tilde{\mathcal{B}}^\infty} \varphi_s\,\mathbf{B}_a\cdot\mathbf{n}'\,\mathrm{d}a - \frac{1}{2}\int_{\partial\mathcal{B}^\infty\cup\partial\tilde{\mathcal{B}}^\infty} \varphi_a\,\mathbf{B}_a\cdot\mathbf{n}'\,\mathrm{d}a - \frac{1}{2}\int_{\partial\mathcal{B}^\infty\cup\partial\tilde{\mathcal{B}}^\infty} \varphi_s\,\mathbf{B}_s\cdot\mathbf{n}'\,\mathrm{d}a. \tag{33}$$

The second integral in (33) is not affected by any variation [14], so we can omit it from our formulation. In the third integral in (33), as in [14], we assume that $\varphi_s \sim 1/|\mathbf{x}|$ as $|\mathbf{x}| \to \infty$. For $\partial\mathcal{B}^\infty$ it follows that $|\mathbf{B}_s| \sim 1/|\mathbf{x}|^2$, and the associated integral vanishes. We assume that the behaviour of $\mathbf{B}_s$ is such that the integral over $\partial\tilde{\mathcal{B}}^\infty$ also vanishes. For the first integral in (33) we can replace φ_s by φ (since the difference is a constant). Hence, equation (32) can be written as

$$E = \int_{\mathcal{B}} \rho\psi^* \,\mathrm{d}v + \int_{\tilde{\mathcal{B}}} \tilde{\rho}\tilde{\psi}^* \,\mathrm{d}v - \frac{1}{2}\mu_0 \int_{\mathcal{B}'} \mathbf{H}\cdot\mathbf{H}\,\mathrm{d}v - \int_{\partial\mathcal{B}^\infty} \varphi\,\mathbf{B}_a\cdot\mathbf{n}'\,\mathrm{d}a - \int_{\partial\tilde{\mathcal{B}}^\infty} \varphi\,\mathbf{B}_a\cdot\tilde{\mathbf{n}}\,\mathrm{d}a. \tag{34}$$

For the mechanical boundary conditions we have $\mathbf{x} = \hat{\mathbf{x}}$ on $\partial\mathcal{B}^\beta$. Also, we have $\dot{\mathbf{x}} = \mathbf{0}$ on $\partial\mathcal{B}^\beta$ (and on $\mathcal{S}$). On $\partial\mathcal{B}^\alpha$ there is no mechanical load or restriction on the displacement. We assume, as in [14], that the mechanical body force $\mathbf{f}$ is conservative and equal to $-\mathrm{grad}\,U$, where U is the associated potential. Then, since there is no prescribed mechanical traction on $\partial\mathcal{B}^\alpha$ and $\mathbf{x}$ is prescribed on $\partial\mathcal{B}^\beta$, the work L of the mechanical loading is simply

$$L = -\int_{\mathcal{B}} \rho U \,\mathrm{d}v. \tag{35}$$

We define the functional Π as[3]

$$\Pi\{\mathbf{x},\varphi\} = E\{\mathbf{x},\varphi\} - L\{\mathbf{x}\}. \tag{36}$$

4.1.1 Variation in the magnetic potential

Let $\dot{\Pi}_\varphi$ denote the variation of the functional Π with respect to the variation $\dot{\varphi}$ in φ. From (34) and the definition (20) we then have

$$\begin{aligned}\dot{\Pi}_\varphi = &-\int_{\mathcal{B}} \rho\frac{\partial\psi^*}{\partial\mathbf{H}}\cdot\mathrm{grad}\,\dot{\varphi}\,\mathrm{d}v - \int_{\tilde{\mathcal{B}}} \tilde{\rho}\frac{\partial\tilde{\psi}^*}{\partial\mathbf{H}}\cdot\mathrm{grad}\,\dot{\varphi}\,\mathrm{d}v + \mu_0\int_{\mathcal{B}'} \mathbf{H}\cdot\mathrm{grad}\,\dot{\varphi}\,\mathrm{d}v - \int_{\partial\mathcal{B}^\infty} \dot{\varphi}\,\mathbf{B}_a\cdot\mathbf{n}'\,\mathrm{d}a \\ &- \int_{\partial\tilde{\mathcal{B}}^\infty} \dot{\varphi}\,\mathbf{B}_a\cdot\tilde{\mathbf{n}}\,\mathrm{d}a.\end{aligned} \tag{37}$$

Following Bustamante et al. [14] we consider the connection

$$-\rho\frac{\partial\psi^*}{\partial\mathbf{H}} = \mathbf{B} \tag{38}$$

in $\mathcal{B}$, and we also write

$$-\tilde{\rho}\frac{\partial\tilde{\psi}^*}{\partial\mathbf{H}} = \mathbf{B} \tag{39}$$

in $\tilde{\mathcal{B}}$. Then, by using $(4)_2$, the divergence theorem, and the decomposition of the boundary $\partial\mathcal{B}$, the first integral in (37) is seen to be equivalent to

$$\int_{\partial\mathcal{B}^\alpha} \dot{\varphi}\,\mathbf{B}\cdot\mathbf{n}\,\mathrm{d}a + \int_{\partial\mathcal{B}^\beta} \dot{\varphi}\,\mathbf{B}\cdot\mathbf{n}\,\mathrm{d}a - \int_{\mathcal{B}} \dot{\varphi}\,\mathrm{div}\,\mathbf{B}\,\mathrm{d}v. \tag{40}$$

Similar expressions can be found for the second and third integrals in (37), but for brevity we do not include them here. Taking account of the decompositions $\partial\tilde{\mathcal{B}} = \partial\mathcal{B}^\beta \cup \mathcal{S} \cup \partial\tilde{\mathcal{B}}^\infty$ and $\partial\mathcal{B}' = \partial\mathcal{B}^\alpha \cup \mathcal{S} \cup \partial\mathcal{B}^\infty$ (see Fig. 4), and remembering the rule for the sign of the normal vectors, we can show that (37) can be written as

$$\begin{aligned}\dot{\Pi}_\varphi = &\int_{\mathcal{B}\cup\tilde{\mathcal{B}}\cup\mathcal{B}'} \dot{\varphi}\,\mathrm{div}\,\mathbf{B}\,\mathrm{d}v + \int_{\partial\mathcal{B}^\alpha} \dot{\varphi}\,[\![\mathbf{B}]\!]\cdot\mathbf{n}\,\mathrm{d}a + \int_{\partial\mathcal{B}^\beta} \dot{\varphi}\,[\![\mathbf{B}]\!]\cdot\mathbf{n}\,\mathrm{d}a \\ &+ \int_{\mathcal{S}} \dot{\varphi}\,[\![\mathbf{B}]\!]\cdot\tilde{\mathbf{n}}\,\mathrm{d}a - \int_{\partial\mathcal{B}^\infty} \dot{\varphi}\,\mathbf{B}_s\cdot\mathbf{n}'\,\mathrm{d}a - \int_{\partial\tilde{\mathcal{B}}^\infty} \dot{\varphi}\,\mathbf{B}_s\cdot\tilde{\mathbf{n}}\,\mathrm{d}a.\end{aligned} \tag{41}$$

[3] A different method to find variational formulations in electro–elasticity has been proposed by He and coworkers, see, for example, [18,40–42].

In the above expression the last two integrals vanish when $|\mathbf{x}| \to \infty$. It follows that Π is stationary with respect to φ if and only if

$$\operatorname{div} \mathbf{B} = 0 \quad \text{in} \;\; \mathcal{B} \cup \tilde{\mathcal{B}} \cup \mathcal{B}' \tag{42}$$

and

$$[\![\mathbf{B}]\!] \cdot \mathbf{n} = 0 \quad \text{on} \;\; \partial\mathcal{B}^{\alpha}, \tag{43}$$

$$[\![\mathbf{B}]\!] \cdot \mathbf{n} = 0 \quad \text{on} \;\; \partial\mathcal{B}^{\beta}, \tag{44}$$

$$[\![\mathbf{B}]\!] \cdot \tilde{\mathbf{n}} = 0 \quad \text{on} \;\; \mathcal{S}. \tag{45}$$

4.1.2 Variation with $\mathbf{x}$

From [21] we have the following connection between the function ψ^* and the complementary form of the energy function Ω^* [3,4]

$$J^{-1}\Omega^*(\mathbf{F}, \mathbf{H}_l) = \rho\psi^*(\mathbf{F}, \mathbf{H}). \tag{46}$$

Let us rewrite some of the integrals in (34) with respect to the reference configuration, we have

$$E = \int_{\mathcal{B}_r} \Omega^* \, \mathrm{d}V + \int_{\tilde{\mathcal{B}}} \tilde{\rho}\tilde{\psi}^* \, \mathrm{d}v - \frac{1}{2}\mu_0 \int_{\mathcal{B}'_r} J(\mathbf{F}^{-T}\mathbf{H}_l) \cdot (\mathbf{F}^{-T}\mathbf{H}_l) \, \mathrm{d}V \\ - \int_{\partial\mathcal{B}^{\infty}} \varphi \, \mathbf{B}_a \cdot \mathbf{n}' \, \mathrm{d}a - \int_{\partial\tilde{\mathcal{B}}^{\infty}} \varphi \, \mathbf{B}_a \cdot \tilde{\mathbf{n}} \, \mathrm{d}a. \tag{47}$$

We did not have to modify the expression for the second, fourth or fifth integrals because the body $\tilde{\mathcal{B}}$ is rigid and $\partial\mathcal{B}^{\infty}$ and $\partial\tilde{\mathcal{B}}^{\infty}$ are assumed fixed. In $\mathcal{B}'_r$ the deformation gradient is ‘fictitious’ in the sense that it is obtained arbitrarily from a smooth extension of the deformation $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ in $\mathcal{B}_r$ to $\mathcal{B}'_r$ (see [22]).

Next, we calculate the variation of E with respect to $\mathbf{x}$, which we denote by $\dot{E}_{\mathbf{x}}$. For this purpose, we use the connection [3,4]

$$\frac{\partial\Omega^*}{\partial\mathbf{F}} = \mathbf{T}, \tag{48}$$

where $\mathbf{T}$ is the total nominal stress, and recall that $\tilde{\psi}^*$ is a function of $\mathbf{H}$ only. We also use the definition [33,36]

$$\boldsymbol{\tau}_m = \mathbf{B} \otimes \mathbf{H} - \frac{1}{2}(\mathbf{B} \cdot \mathbf{H})\mathbf{I} \tag{49}$$

of the Maxwell stress.

Following a procedure similar to that used in [14], the variation of E with respect to $\mathbf{x}$ can be written as

$$\dot{E}_{\mathbf{x}} = \int_{\mathcal{B}_r} \operatorname{tr}(\mathbf{T}\dot{\mathbf{F}}) \, \mathrm{d}V + \int_{\mathcal{B}'} \operatorname{div}(\boldsymbol{\tau}_m\dot{\mathbf{x}}) \, \mathrm{d}v - \int_{\mathcal{B}'} (\operatorname{div}\boldsymbol{\tau}_m) \cdot \dot{\mathbf{x}} \, \mathrm{d}v. \tag{50}$$

Remark 1 We must be aware that since $\varphi = \varphi(\mathbf{x}) = \varphi(\boldsymbol{\chi}(\mathbf{X}))$, the variation in $\mathbf{x}$ induces a variation in φ, which we can denote $\dot{\varphi}_{\text{ind}}$. In this case we have $\dot{\mathbf{H}}_{\text{ind}} = -\operatorname{grad}\dot{\varphi}_{\text{ind}}$ as the induced magnetic field. It is possible to prove that, if we include the induced variation in φ, we can obtain the same set of integrals shown in (41), but with $\dot{\varphi}_{\text{ind}}$ in the place of $\dot{\varphi}$. From (42–45) we have that these integrals are zero and this is why $\dot{\varphi}_{\text{ind}}$ does not appear in (50). For this reason we do not consider the induced variation in the subsequent formulations.

On using the divergence theorem in (50), making some rearrangements, referring to (35), and noting that $\dot{\mathbf{x}} = \mathbf{0}$ on $\mathcal{S}$ and $\partial\mathcal{B}^{\beta}$, we obtain

$$\dot{\Pi}_{\mathbf{x}} = -\int_{\mathcal{B}_r} (\operatorname{Div}\mathbf{T} + \rho_r\mathbf{f}) \cdot \dot{\mathbf{x}} \, \mathrm{d}V - \int_{\mathcal{B}'} (\operatorname{div}\boldsymbol{\tau}_m) \cdot \dot{\mathbf{x}} \, \mathrm{d}v + \int_{\partial\mathcal{B}_r^{\alpha}} (\mathbf{T}^T\mathbf{N} - \mathbf{t}_m) \cdot \dot{\mathbf{x}} \, \mathrm{d}A + \int_{\partial\mathcal{B}^{\infty}} \boldsymbol{\tau}_m\mathbf{n}' \cdot \dot{\mathbf{x}} \, \mathrm{d}a, \tag{51}$$

where $\rho_r = \rho J$ is the density in $\mathcal{B}_r$, $\mathbf{t}_M = \mathbf{T}_M^T\mathbf{N}$ and $\mathbf{T}_M = J\mathbf{F}^{-1}\boldsymbol{\tau}_m$.

In the above equation we can assume that the fourth integral vanishes for $|\mathbf{x}| \to \infty$. As a result Π is stationary with respect to $\mathbf{x}$ if and only if

$$\mathrm{Div}\,\mathbf{T} + \rho_r \mathbf{f} = \mathbf{0} \quad \text{in } \mathcal{B}_r, \tag{52}$$

$$\mathbf{T}^T\mathbf{N} = \mathbf{t}_M \quad \text{on } \partial\mathcal{B}_r^{\alpha} \tag{53}$$

and

$$\mathrm{div}\,\boldsymbol{\tau}_m = \mathbf{0} \quad \text{in } \mathcal{B}'. \tag{54}$$

Note that this last equation holds for free space if the field equations (4) are satisfied.

To summarize, the functional Π is stationary with respect to both φ and $\mathbf{x}$ if and only if Eqs. (42–45), (52) and (53) hold.

4.2 Interaction of a MS elastomer with a rigid semi-infinite body. Formulation based on the vector magnetic potential

In (34) we see the use of the applied (far away) magnetic induction $\mathbf{B}_a$, which means we were assuming that far away there was source of magnetic induction (such as an electric current in an electromagnet). Let us assume now there is a far-away source of magnetic field $\mathbf{H}_a$. Consider the following expression for the energy of the system [14,39]

$$E^*\{\mathbf{x}, \mathbf{A}\} = \int_{\mathcal{B}\cup\tilde{\mathcal{B}}} \varrho\Upsilon^*\,\mathrm{d}v - \frac{1}{2}\int_{\mathcal{B}\cup\tilde{\mathcal{B}}} \mathbf{B}\cdot\mathbf{H}\,\mathrm{d}v - \frac{1}{2}\mu_0\int_{\mathcal{B}\cup\tilde{\mathcal{B}}} \mathbf{M}\cdot\mathbf{H}_a\,\mathrm{d}v. \tag{55}$$

A solution of the field equation $(4)_2$ is given by

$$\mathbf{B} = \mathrm{curl}\,\mathbf{A}, \tag{56}$$

where $\mathbf{A}$ is known as the magnetic vector potential. We can decompose the vector potential in an applied and a self part, and in such a case we have

$$\mathbf{B}_a = \mathrm{curl}\,\mathbf{A}_a \quad \text{and} \quad \mathbf{B}_s = \mathrm{curl}\,\mathbf{A}_s. \tag{57}$$

The energy function Υ^* is defined as

$$\Upsilon^* = \begin{cases} \phi^* & \mathbf{x}\in\mathcal{B}, \\ \tilde{\phi}^* & \mathbf{x}\in\tilde{\mathcal{B}}, \end{cases} \tag{58}$$

where [14] $\phi^* = \phi^*(\mathbf{F}, \mathbf{B})$ and $\tilde{\phi}^* = \tilde{\phi}^*(\mathbf{B})$, and ϱ has been defined in (24).

The first integral of the right side of (55) can be decomposed as

$$\int_{\mathcal{B}} \rho\phi^*\,\mathrm{d}v + \int_{\tilde{\mathcal{B}}} \tilde{\rho}\tilde{\phi}^*\,\mathrm{d}v. \tag{59}$$

Consider the identity

$$\mathrm{curl}\,\mathbf{A}\cdot\mathbf{H} = \mathrm{div}\,(\mathbf{H}\times\mathbf{A}) + \mathbf{A}\cdot\mathrm{curl}\,\mathbf{H}. \tag{60}$$

Regarding the second integral on the right side of (55), using (60), Eq. 4_1 and remembering Fig. 4 we have

$$-\frac{1}{2}\int_{\mathcal{B}\cup\tilde{\mathcal{B}}} \mathbf{B}\cdot\mathbf{H}\,\mathrm{d}v = -\frac{1}{2}\int_{\partial\mathcal{B}^{\alpha}\cup\mathcal{S}} (\mathbf{H}\times\mathbf{A})\cdot\mathbf{n}\,\mathrm{d}a - \frac{1}{2}\int_{\partial\tilde{\mathcal{B}}^{\infty}} (\mathbf{H}\times\mathbf{A})\cdot\mathbf{n}\,\mathrm{d}a, \tag{61}$$

where for the first integral of the right side of (61), using (56), the divergence theorem and from Fig. 4 we have

$$-\frac{1}{2}\int_{\partial\mathcal{B}^{\alpha}\cup\mathcal{S}} (\mathbf{H}\times\mathbf{A})\cdot\mathbf{n}\,\mathrm{d}a = \frac{1}{2}\int_{\mathcal{B}'} \mathrm{curl}\,\mathbf{A}\cdot\mathbf{H}\,\mathrm{d}v - \frac{1}{2}\int_{\partial\mathcal{B}^{\infty}} (\mathbf{H}\times\mathbf{A})\cdot\mathbf{n}'\,\mathrm{d}a. \tag{62}$$

For the third integral in (55), using $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$ we get

$$-\frac{1}{2}\mu_0 \int_{\mathcal{B}\cup\tilde{\mathcal{B}}} \mathbf{M}\cdot\mathbf{H}_a \,\mathrm{d}v = -\frac{1}{2}\int_{\mathcal{B}\cup\tilde{\mathcal{B}}} \operatorname{curl}\mathbf{A}\cdot\mathbf{H}_a \,\mathrm{d}v + \frac{1}{2}\int_{\mathcal{B}\cup\tilde{\mathcal{B}}} \mathbf{H}\cdot\operatorname{curl}\mathbf{A}_a \,\mathrm{d}v, \tag{63}$$

and using (60), the divergence theorem and $(4)_1$ we can prove that

$$-\frac{1}{2}\int_{\mathcal{B}\cup\tilde{\mathcal{B}}} \operatorname{curl}\mathbf{A}\cdot\mathbf{H}_a \,\mathrm{d}v = \frac{1}{2}\int_{\mathcal{B}'} \operatorname{curl}\mathbf{A}\cdot\mathbf{H}_a \,\mathrm{d}v - \frac{1}{2}\int_{\partial\mathcal{B}^\infty} (\mathbf{H}_a\times\mathbf{A})\cdot\mathbf{n}' \,\mathrm{d}a - \frac{1}{2}\int_{\partial\tilde{\mathcal{B}}^\infty} (\mathbf{H}_a\times\mathbf{A})\cdot\tilde{\mathbf{n}} \,\mathrm{d}a, \tag{64}$$

and

$$\frac{1}{2}\int_{\mathcal{B}\cup\tilde{\mathcal{B}}} \operatorname{curl}\mathbf{A}_a\cdot\mathbf{H} \,\mathrm{d}v = -\frac{1}{2}\int_{\mathcal{B}'} \operatorname{curl}\mathbf{A}_a\cdot\mathbf{H} \,\mathrm{d}v + \frac{1}{2}\int_{\partial\mathcal{B}^\infty} (\mathbf{H}\times\mathbf{A}_a)\cdot\mathbf{n}' \,\mathrm{d}a + \frac{1}{2}\int_{\partial\tilde{\mathcal{B}}^\infty} (\mathbf{H}\times\mathbf{A}_a)\cdot\tilde{\mathbf{n}} \,\mathrm{d}a. \tag{65}$$

Using (64) and (65) in (63), and (61–63) in (55), assuming that $\int_{\partial\mathcal{B}^\infty}(\mathbf{H}_s\times\mathbf{A}_s)\cdot\mathbf{n}\,\mathrm{d}a \to 0$ and $\int_{\partial\tilde{\mathcal{B}}^\infty}(\mathbf{H}_s\times\mathbf{A}_s)\cdot\mathbf{n}\,\mathrm{d}a \to 0$ as $|\mathbf{x}|\to\infty$ (see [14]) we finally get

$$E^*\{\mathbf{x},\mathbf{A}\} = \int_{\mathcal{B}} \rho\phi^* \,\mathrm{d}v + \int_{\tilde{\mathcal{B}}} \tilde{\rho}\tilde{\phi}^* \,\mathrm{d}v + \frac{1}{2\mu_0}\int_{\mathcal{B}'} \mathbf{B}\cdot\mathbf{B} \,\mathrm{d}v - \int_{\partial\mathcal{B}^\infty} (\mathbf{H}_a\times\mathbf{A})\cdot\mathbf{n}' \,\mathrm{d}a - \int_{\partial\tilde{\mathcal{B}}^\infty} (\mathbf{H}_a\times\mathbf{A})\cdot\tilde{\mathbf{n}} \,\mathrm{d}a. \tag{66}$$

We can now define the functional $\Pi^* = \Pi^*\{\mathbf{x},\mathbf{A}\}$ as

$$\Pi^*\{\mathbf{x},\mathbf{A}\} = E^*\{\mathbf{x},\mathbf{A}\} - L\{\mathbf{x}\}. \tag{67}$$

Let us calculate the first variation of the above functional in terms of the vector potential; we have

$$\begin{aligned}\dot{\Pi}^*_{\mathbf{A}} = {} & \int_{\mathcal{B}} \rho\frac{\partial\phi^*}{\partial\mathbf{B}}\cdot\operatorname{curl}\dot{\mathbf{A}} \,\mathrm{d}v + \int_{\tilde{\mathcal{B}}} \tilde{\rho}\frac{\partial\tilde{\phi}^*}{\partial\mathbf{B}}\cdot\operatorname{curl}\dot{\mathbf{A}} \,\mathrm{d}v + \frac{1}{\mu_0}\int_{\mathcal{B}'} \mathbf{B}\cdot\operatorname{curl}\dot{\mathbf{A}} \,\mathrm{d}v - \int_{\partial\mathcal{B}^\infty} (\mathbf{H}_a\times\dot{\mathbf{A}})\cdot\mathbf{n}' \,\mathrm{d}a \\ & - \int_{\partial\tilde{\mathcal{B}}^\infty} (\mathbf{H}_a\times\dot{\mathbf{A}})\cdot\tilde{\mathbf{n}} \,\mathrm{d}a, \end{aligned} \tag{68}$$

but from [14] we have that

$$\rho\frac{\partial\phi^*}{\partial\mathbf{B}} = \mathbf{H}, \quad \text{in } \mathcal{B}, \tag{69}$$

and we write

$$\tilde{\rho}\frac{\partial\tilde{\phi}^*}{\partial\mathbf{B}} = \mathbf{H}, \quad \text{in } \tilde{\mathcal{B}}. \tag{70}$$

With (69) and (70), using the identity $\operatorname{curl}\dot{\mathbf{A}}\cdot\mathbf{H} = \operatorname{div}(\mathbf{H}\times\dot{\mathbf{A}}) + \dot{\mathbf{A}}\cdot\operatorname{curl}\mathbf{H}$, considering the divergence theorem and recalling Fig. 4 we can prove that

$$\begin{aligned}\dot{\Pi}^*_{\mathbf{A}} = {} & \int_{\mathcal{B}\cup\tilde{\mathcal{B}}\cup\mathcal{B}'} \dot{\mathbf{A}}\cdot\operatorname{curl}\mathbf{H} \,\mathrm{d}v + \int_{\partial\mathcal{B}^\alpha} (\mathbf{n}\times[\![\mathbf{H}]\!])\cdot\dot{\mathbf{A}} \,\mathrm{d}a + \int_{\partial\mathcal{B}^\beta} (\mathbf{n}\times[\![\mathbf{H}]\!])\cdot\dot{\mathbf{A}} \,\mathrm{d}a \\ & + \int_{\mathcal{S}} (\mathbf{n}\times[\![\mathbf{H}]\!])\cdot\dot{\mathbf{A}} \,\mathrm{d}a + \int_{\partial\mathcal{B}^\infty} [(\mathbf{H}-\mathbf{H}_a)\times\dot{\mathbf{A}}]\cdot\mathbf{n}' \,\mathrm{d}a + \int_{\partial\tilde{\mathcal{B}}^\infty} [(\mathbf{H}-\mathbf{H}_a)\times\dot{\mathbf{A}}]\cdot\mathbf{n}' \,\mathrm{d}a, \end{aligned} \tag{71}$$

but $\mathbf{H}_s = \mathbf{H} - \mathbf{H}_a$, and we have that $\int_{\partial\mathcal{B}^\infty}(\mathbf{H}_s\times\dot{\mathbf{A}}_s)\cdot\mathbf{n}\,\mathrm{d}a \to 0$ and $\int_{\partial\tilde{\mathcal{B}}^\infty}(\mathbf{H}_s\times\dot{\mathbf{A}}_s)\cdot\mathbf{n}\,\mathrm{d}a \to 0$ as $|\mathbf{x}|\to\infty$, and from (71) it follows that Π^* is stationary if and only if

$$\operatorname{curl}\mathbf{H} = \mathbf{0} \quad \text{in } \mathbf{x}\in\mathcal{B}\cup\tilde{\mathcal{B}}\cup\mathcal{B}', \tag{72}$$

and

$$\mathbf{n}\times[\![\mathbf{H}]\!] = \mathbf{0} \quad \text{on } \mathbf{x}\in\partial\mathcal{B}^\alpha, \tag{73}$$

$$\mathbf{n}\times[\![\mathbf{H}]\!] = \mathbf{0} \quad \text{on } \mathbf{x}\in\partial\mathcal{B}^\beta, \tag{74}$$

$$\mathbf{n}\times[\![\mathbf{H}]\!] = \mathbf{0} \quad \text{on } \mathbf{x}\in\partial\mathcal{S}. \tag{75}$$

Regarding the variation of Π^* on $\mathbf{x}$, the expressions that appear are closely similar to the expressions presented in the previous subsection, therefore, for brevity we do not repeat them here (see also [14]).

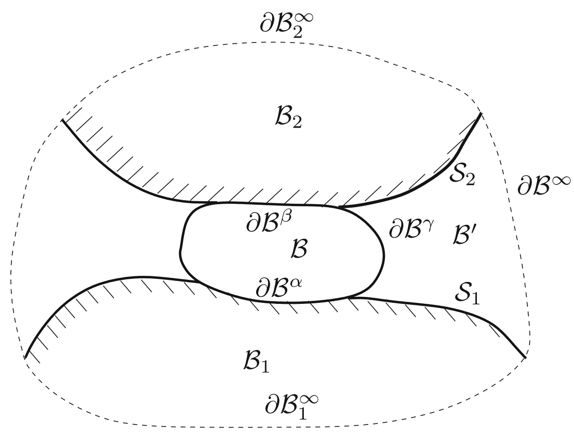

Fig. 5 A mixed-boundary-value problem. A MS elastomer attached to two semi-infinite bodies

4.3 Interaction of a MS elastomer with two semi-infinite bodies

The two models presented previously could be used to work, for example, with the problems shown in Fig. 3a, b. In the case of Fig. 3a the rigid semi-infinite body would be divided into two (disjoint) parts, the inferior that would remain fixed and a superior that would move in the axial direction (blue arrow).

We would like to explore a model where we could prescribe some sort of *external mechanical load*. Consider Fig. 5 where we have a MS elastomeric body $\mathcal{B}$ attached to two semi-infinite bodies $\mathcal{B}_1$ and $\mathcal{B}_2$. The body $\mathcal{B}$ is perfectly bonded to $\mathcal{B}_1$ on the surface $\partial\mathcal{B}^\alpha$ and to $\mathcal{B}_2$ on the surface $\partial\mathcal{B}^\beta$. Surrounding the bodies $\mathcal{B}$, $\mathcal{B}_1$ and $\mathcal{B}_2$ we have a free (vacuum) space $\mathcal{B}'$; the surface that separates $\mathcal{B}'$ and $\mathcal{B}$ is denoted by $\partial\mathcal{B}^\gamma$, and the surfaces that separate $\mathcal{B}'$ with $\mathcal{B}_1$ and $\mathcal{B}_2$ are denoted by $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. The boundary of the free space far away is denoted by $\partial\mathcal{B}^\infty$, and for the semi-infinite bodies $\mathcal{B}_1$ and $\mathcal{B}_2$ the boundaries far away are denoted by $\partial\mathcal{B}_1^\infty$ and $\partial\mathcal{B}_2^\infty$, respectively.

We assume that the body $\mathcal{B}_1$ is rigid and magnetizable. Far away inside the body there is an applied magnetic induction. As for the boundary of the free space far away, we assume there is an applied induction field as well. This body could move and rotate rigidly.

Regarding the body $\mathcal{B}_2$ we assume that this body is elastic and magnetizable. Far away in $\partial\mathcal{B}_2^\infty$ a mechanical surface load and a magnetic induction are applied.[4] The densities of the bodies $\mathcal{B}$, $\mathcal{B}_1$ and $\mathcal{B}_2$ are denoted ρ, ρ_1 and ρ_2, respectively.

We work with the magnetic field as the independent magnetic variable, and as we did in Sect. 4.1 we assume there exists a scalar potential φ such that $\mathbf{H} = -\mathrm{grad}\,\varphi$. For the MS body $\mathcal{B}$ we assume there exists an energy function $\psi^* = \psi^*(\mathbf{F}, \mathbf{H})$ such that (see [14])

$$\mathbf{B} = -\rho \frac{\partial \psi^*}{\partial \mathbf{H}}, \tag{76}$$

which is related with the total energy function of Dorfmann and Ogden [3,4] by

$$J^{-1}\Omega^*(\mathbf{F}, \mathbf{H}_l) = \rho\psi^*(\mathbf{F}, \mathbf{H}), \tag{77}$$

and so we have

$$\mathbf{T} = \frac{\partial \Omega^*}{\partial \mathbf{F}}. \tag{78}$$

The body $\mathcal{B}_1$ is rigid and magnetizable and we assume there exists an energy function $\psi_1^* = \psi_1^*(\mathbf{H})$ such that

$$\mathbf{B} = -\rho_1 \frac{\partial \psi_1^*}{\partial \mathbf{H}}. \tag{79}$$

This body may move and rotate rigidly.

4 We do not discuss the possibility of adding Maxwell stresses to this external surface load (see [33]).

As for the body $\mathcal{B}_2$ we assume this is deformable and magnetizable; as well as this, we assume there exists an energy function $\psi_2^* = \psi_2^*(\mathbf{F}, \mathbf{H})$ such that

$$\mathbf{B} = -\rho_2 \frac{\partial \psi_2^*}{\partial \mathbf{H}}, \tag{80}$$

and

$$\mathbf{T} = \frac{\partial \Omega_2^*}{\partial \mathbf{F}}, \tag{81}$$

where we have defined

$$J^{-1}\Omega_2^*(\mathbf{F}, \mathbf{H}_l) = \rho_2 \psi_2^*(\mathbf{F}, \mathbf{H}). \tag{82}$$

Remark 2 We could assume very simple forms for $\psi_1^*(\mathbf{H})$ and $\psi_2^*(\mathbf{F}, \mathbf{H})$, such that the constitutive equations for the magnetic induction and the stress would be linear expressions; see, for example, [17]. We do not explore further simplifications in this paper and we assume $\psi_1^*(\mathbf{H})$ and $\psi_2^*(\mathbf{F}, \mathbf{H})$ as general as possible.

From the previous two subsections we have a number of results, which we do not need to repeat here. Consider the functional

$$\Xi\{\mathbf{x}, \varphi\} = E\{\mathbf{x}, \varphi\} - L\{\mathbf{x}\}, \tag{83}$$

where in this model the energy functional E is given as

$$\begin{aligned} E\{\mathbf{x}, \varphi\} = &\int_{\mathcal{B}} \rho \psi^* \,\mathrm{d}v + \int_{\mathcal{B}_1} \rho_1 \psi_1^* \,\mathrm{d}v + \int_{\mathcal{B}_2} \rho_2 \psi_2^* \,\mathrm{d}v - \frac{1}{2}\mu_0 \int_{\mathcal{B}'} \mathbf{H}\cdot\mathbf{H}\,\mathrm{d}v \\ &- \int_{\partial\mathcal{B}^\infty} \varphi \mathbf{B}_a \cdot \mathbf{n}' \,\mathrm{d}a - \int_{\partial\mathcal{B}_1^\infty} \varphi \mathbf{B}_a \cdot \mathbf{n}_1 \,\mathrm{d}a - \int_{\partial\mathcal{B}_2^\infty} \varphi \mathbf{B}_a \cdot \mathbf{n}_2 \,\mathrm{d}a; \end{aligned} \tag{84}$$

here $\mathbf{n}_1$ and $\mathbf{n}_2$ are the normal unit outward vectors to the surfaces $\partial\mathcal{B}_1^\infty$ and $\partial\mathcal{B}_2^\infty$, respectively.

In this problem the functional $L\{\mathbf{x}\}$ is defined as

$$L\{\mathbf{x}\} = -\int_{\mathcal{B}} \rho U \,\mathrm{d}v - \int_{\mathcal{B}_2} \rho U_2 \,\mathrm{d}v + \int_{\partial\mathcal{B}_2^\infty} \mathbf{x}\cdot\mathbf{t}_a \,\mathrm{d}a, \tag{85}$$

where we have assumed that the body forces in $\mathcal{B}$ and $\mathcal{B}_2$ can be expressed in term of the gradient of the potentials U and U_2 respectively, i.e.,

$$\mathbf{f} = -\rho \operatorname{grad} U, \quad \text{and} \quad \mathbf{f}_2 = -\rho_2 \operatorname{grad} U_2. \tag{86}$$

The term $\mathbf{t}_a$ corresponds to the mechanical surface load applied far away on $\partial\mathcal{B}_2^\infty$.

4.3.1 Variation on the scalar magnetic potential

It is easy to prove that the first variation of Ξ is given as (see Sect. 4.1 and [14,34])

$$\begin{aligned} \dot{\Xi}_\varphi = &\int_{\mathcal{B}\cup\mathcal{B}_1\cup\mathcal{B}_2\cup\mathcal{B}'} \dot{\varphi} \operatorname{div}\mathbf{B}\,\mathrm{d}v + \int_{\partial\mathcal{B}^\alpha} \dot{\varphi}\, [\![\mathbf{B}]\!]\cdot\mathbf{n}\,\mathrm{d}a + \int_{\partial\mathcal{B}^\beta} \dot{\varphi}\, [\![\mathbf{B}]\!]\cdot\mathbf{n}\,\mathrm{d}a + \int_{\mathcal{S}_1} \dot{\varphi}\, [\![\mathbf{B}]\!]\cdot\mathbf{n}_1\,\mathrm{d}a \\ &+ \int_{\mathcal{S}_2} \dot{\varphi}\, [\![\mathbf{B}]\!]\cdot\mathbf{n}_2\,\mathrm{d}a - \int_{\partial\mathcal{B}^\infty} \dot{\varphi}\,\mathbf{B}_s\cdot\mathbf{n}'\,\mathrm{d}a - \int_{\partial\mathcal{B}_1^\infty} \dot{\varphi}\,\mathbf{B}_s\cdot\mathbf{n}_1\,\mathrm{d}a - \int_{\partial\mathcal{B}_2^\infty} \dot{\varphi}\,\mathbf{B}_s\cdot\mathbf{n}_2\,\mathrm{d}a. \end{aligned} \tag{87}$$

The variation of the potential φ far away only corresponds to the variation of φ_s, so by a standard argument [14,34] we assume that the surface integrals of $\dot{\varphi}\,\mathbf{B}_s\cdot\mathbf{n}_1$, $\dot{\varphi}\,\mathbf{B}_s\cdot\mathbf{n}_2$ and $\dot{\varphi}\,\mathbf{B}_s\cdot\mathbf{n}'$ vanish. Therefore Ξ is stationary with respect to φ if and only if

$$\operatorname{div}\mathbf{B} = 0 \quad \text{in} \;\; \mathcal{B}\cup\mathcal{B}_1\cup\mathcal{B}_2\cup\mathcal{B}' \tag{88}$$

and

$$[\![\mathbf{B}]\!]\cdot\mathbf{n} = 0 \quad \text{on} \;\; \partial\mathcal{B}^\alpha, \tag{89}$$

$$[\![\mathbf{B}]\!]\cdot\mathbf{n} = 0 \quad \text{on} \; \partial\mathcal{B}^\beta, \tag{90}$$

$$[\![\mathbf{B}]\!]\cdot\mathbf{n}_1 = 0 \quad \text{on} \;\; \mathcal{S}_1. \tag{91}$$

$$[\![\mathbf{B}]\!]\cdot\mathbf{n}_2 = 0 \quad \text{on} \;\; \mathcal{S}_2. \tag{92}$$

4.3.2 Variation with $\mathbf{x}$

To calculate the variation of Ξ in terms of $\mathbf{x}$, let us rewrite some of the integrals in the functionals E and L in the reference configuration [14]. Using the connections (77) and (82) we have

$$E = \int_{\mathcal{B}_r} \Omega^* \,\mathrm{d}V + \int_{\mathcal{B}_1} \rho_1 \psi_1^* \,\mathrm{d}v + \int_{\mathcal{B}_{r_2}} \Omega_2^* \,\mathrm{d}V - \frac{1}{2}\mu_0 \int_{\mathcal{B}'_r} J(\mathbf{F}^{-\mathrm{T}}\mathbf{H}_l)\cdot(\mathbf{F}^{-\mathrm{T}}\mathbf{H}_l)\,\mathrm{d}V$$
$$- \int_{\partial\mathcal{B}^\infty} \varphi \mathbf{B}_a \cdot \mathbf{n}' \,\mathrm{d}a - \int_{\partial\mathcal{B}_1^\infty} \varphi \mathbf{B}_a \cdot \mathbf{n}_1 \,\mathrm{d}a - \int_{\partial\mathcal{B}_{r_2}^\infty} \varphi \mathbf{B}_{l_a} \cdot \mathbf{N}_2 \,\mathrm{d}A, \tag{93}$$

where $\mathcal{B}_r$, $\mathcal{B}_{r_2}$ and $\partial\mathcal{B}_{r_2}^\infty$ correspond to the reference forms of the bodies $\mathcal{B}$, $\mathcal{B}_2$ and the far-away boundary of $\mathcal{B}_2$, respectively. We have that $\mathbf{B}_{l_a}$ is the Lagrangian form for the applied far-away magnetic induction. We have assumed that $\partial\mathcal{B}^\infty$ and $\partial\mathcal{B}_1^\infty$ remain fixed. In (93) we have extended the displacement field $\mathbf{x}$ to free space [22].

It is easy to prove that the first variation of E in $\mathbf{x}$ is given by

$$\dot{E}_{\mathbf{x}} = \int_{\mathcal{B}_r} \mathrm{tr}(\mathbf{T}\dot{\mathbf{F}})\,\mathrm{d}V + \int_{\mathcal{B}_{r_2}} \mathrm{tr}(\mathbf{T}\dot{\mathbf{F}})\,\mathrm{d}V + \int_{\mathcal{B}'} \mathrm{div}(\boldsymbol{\tau}_m \dot{\mathbf{x}})\,\mathrm{d}v - \int_{\mathcal{B}'} (\mathrm{div}\,\boldsymbol{\tau}_m)\cdot\dot{\mathbf{x}}\,\mathrm{d}v. \tag{94}$$

Using the connection (see [35]) $\mathrm{tr}(\mathbf{T}\dot{\mathbf{F}}) = \mathrm{Div}(\mathbf{T}\dot{\mathbf{x}}) - (\mathrm{Div}\,\mathbf{T})\cdot\dot{\mathbf{x}}$, the divergence theorem, the connections (see Fig. 5)

$$\partial\mathcal{B}_2 = \partial\mathcal{B}^\beta \cup \mathcal{S}_2 \cup \partial\mathcal{B}_2^\infty, \quad \partial\mathcal{B}' = \partial\mathcal{B}^\gamma \cup \mathcal{S}_2 \cup \mathcal{S}_1 \cup \partial\mathcal{B}^\infty, \quad \partial\mathcal{B} = \partial\mathcal{B}^\alpha \cup \partial\mathcal{B}^\beta \cup \partial\mathcal{B}^\gamma, \tag{95}$$

and using $\dot{\mathbf{x}} = \mathbf{0}$ on $\mathcal{S}_1$ and $\partial\mathcal{B}^\alpha$, we can prove that

$$\dot{\Xi}_{\mathbf{x}} = \dot{E}_{\mathbf{x}} - \dot{L}_{\mathbf{x}} = \int_{\partial\mathcal{B}_r^\beta} ([\![\mathbf{T}^\mathrm{T}]\!]\mathbf{N})\cdot\dot{\mathbf{x}}\,\mathrm{d}A + \int_{\partial\mathcal{B}_r^\gamma} (\mathbf{T}^\mathrm{T}\mathbf{N} - \mathbf{t}_M)\cdot\dot{\mathbf{x}}\,\mathrm{d}A + \int_{\mathcal{S}_{r_2}} (\mathbf{T}^\mathrm{T}\mathbf{N}_2 - \mathbf{t}_M)\cdot\dot{\mathbf{x}}\,\mathrm{d}A$$
$$+ \int_{\partial\mathcal{B}_{r_2}^\infty} (\mathbf{T}^\mathrm{T}\mathbf{N}_2 - \mathbf{t}_A)\cdot\dot{\mathbf{x}}\,\mathrm{d}A - \int_{\mathcal{B}_r} (\mathrm{Div}\,\mathbf{T} + \rho_r \mathbf{f})\cdot\dot{\mathbf{x}}\,\mathrm{d}V - \int_{\mathcal{B}_{r_2}} (\mathrm{Div}\,\mathbf{T} + \rho_{r_2}\mathbf{f}_2)\cdot\dot{\mathbf{x}}\,\mathrm{d}V$$
$$+ \int_{\partial\mathcal{B}^\infty} (\boldsymbol{\tau}_m^\mathrm{T}\mathbf{n}')\cdot\dot{\mathbf{x}}\,\mathrm{d}a - \int_{\mathcal{B}'} (\mathrm{div}\,\boldsymbol{\tau}_m)\cdot\dot{\mathbf{x}}\,\mathrm{d}a, \tag{96}$$

where $\boldsymbol{\tau}_m$ is the Maxwell stress tensor and we have defined

$$\mathbf{t}_M \equiv \mathbf{T}_M^\mathrm{T}\mathbf{N}, \quad \text{where } \mathbf{T}_M = J\mathbf{F}^{-1}\boldsymbol{\tau}_m \tag{97}$$

and

$$\mathbf{t}_A\,\mathrm{d}A = \mathbf{t}_a\,\mathrm{d}a. \tag{98}$$

We can assume that $\int_{\partial\mathcal{B}^\infty} (\boldsymbol{\tau}_m^\mathrm{T}\mathbf{n}')\cdot\dot{\mathbf{x}}\,\mathrm{d}a \to 0$ as $|\mathbf{x}| \to \infty$. As well as this, from (4) we can prove that the last integral in (96) vanishes. Therefore Ξ is stationary with respect to $\mathbf{x}$ if and only if

$$\mathrm{Div}\,\mathbf{T} + \rho_r \mathbf{f} = \mathbf{0} \ \text{ in } \mathcal{B}_r, \quad \mathrm{Div}\,\mathbf{T} + \rho_{r_2}\mathbf{f}_2 = \mathbf{0} \ \text{ in } \mathcal{B}_{r_2} \tag{99}$$

and

$$[\![\mathbf{T}^\mathrm{T}]\!]\mathbf{N} = \mathbf{0} \ \text{ on } \partial\mathcal{B}_r^\beta, \tag{100}$$

$$\mathbf{T}^T\mathbf{N} = \mathbf{t}_M \ \text{ on } \partial\mathcal{B}_r^\gamma, \tag{101}$$

$$\mathbf{T}^T\mathbf{N}_2 = \mathbf{t}_M \ \text{ on } \partial\mathcal{S}_{r_2}, \tag{102}$$

$$\mathbf{T}^T\mathbf{N}_2 = \mathbf{t}_A \ \text{ on } \partial\mathcal{B}_{r_2}^\infty. \tag{103}$$

The boundary condition (103) is the important condition we were looking for. Going back to Fig. 3, we can assume a very simple model for the mechanical and magnetic behaviours of the loading devices, and so we could apply some external known force far away on the upper part (blue arrows), along with some external magnetic induction. The lower part of the loading devices (the rigid body) would remain fixed.

5 Final remarks

To arrive at appropriate and simple models for the interaction of a MS elastomer with an external loading device is not simple. We proposed a few models.

Many researchers have only considered the material body, disregarding the effects that may exist due to the surroundings [16,18,24,37]. Of course, as a first step, using such models seems to be a good option; however, in magneto–elasticity we have magnetic and mechanical interactions, and so, to consider a body isolated from the environment implies that we are not only simplifying the boundary conditions for the mechanical part of the problem [28], but also for the electro–magnetic boundary conditions.

In electro– and magneto–statics the basic equations are very similar in form; see, for example, [3–6]. Therefore, most of the variational formulations seen in Sect. 4 can be adapted easily for the electro–elastic problem. Variational formulations for the interaction of an electro-sensitive elastomer and external bodies will be treated in a separate short communication, considering, for example, the case in which electric fields are produced using a distribution of surface charge, and a given electric scalar potential applied from a distance [1].

The variational formulations presented in this paper considered either $\mathbf{x}$, φ or $\mathbf{x}$ and $\mathbf{A}$ as the independent variables. We are also interested in finding 'full' variational formulations, where, for example, $\mathbf{T}$, $\mathbf{F}$, $\mathbf{H}$, etc. were considered as independent variables. In a series of papers He and coworkers [40–42] have used a 'semi-inverse' method to find variational formulations for some complex problems, such as in electro–elastic interactions. That method will be used to generalize the formulations presented here and in [14,15].

We will use some of the models put forward in Sect. 4 to solve some boundary-value problems using the finite-element method.

Finally, in [15] Bustamante et al. studied the case of an ES elastomer surrounded by free space, including the effect of a distribution of free charge on the surface of the body, and a density of charges per unit of volume inside the body. Analogously for MS elastomers we could explore an additional case, in which we would have a distribution of steady electric current per unit of volume $\mathbf{J}$, and a distribution of surface current $\mathbf{K}$. However, for MS elastomers we usually assume that the magneto-active particles are distributed in such a way that each particle is completely surrounded by elastomer, which could be considered essentially as an insulator (from the electrical point of view); so, since the particles would not touch each other, we could not have an electric current per unit of volume. Regarding the distribution of electric surface current, Tiersten [43] claimed that to have a surface current has no physical meaning. As a result we have not considered this extra case in this paper.

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